Huneke-Wiegand conjecture and change of rings

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§1 Introduction

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- M, N finitely generated torsionfree R-modules

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Let R be a commutative Noetherian ring, M a finitely generated R-module. If

$$\mathrm{Ext}_R^i(M, M \oplus R) = (0) \quad \text{for } \forall i > 0,$$

then M is projective.

Huneke-Wiegand conjecture

Let R be a Gorenstein local domain, M a torsionfree R-module. If

$$M \otimes_R \mathrm{Hom}_R(M, R) \quad \text{is reflexive},$$

then M is free.

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Theorem 1.1 ([3, 4, 6])

Consider the following conditions.

(1) (HWC) *holds for Gorenstein local domains.*

(2) (HWC) *holds for one-dimensional Gorenstein local domains.*

(3) (ARC) *holds for Gorenstein local domains.*

Then the implications $(1) \Longleftrightarrow (2) \Longrightarrow (3)$ *hold.*

Conjecture 1.2

Let R be a Gorenstein local domain with $\dim R = 1$ and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

In my talk, we are interested in the question of what happens if we replace $\mathrm{Hom}_R(I, R)$ by $\mathrm{Hom}_R(I, \mathrm{K}_R)$.

Conjecture 1.3

Let R be a C-M local ring with $\dim R = 1$ and assume $\exists \mathrm{K}_R$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree, then $I \cong R$ or K_R as an R-module.

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Advantages

- $\exists$ symmetry between I and $I^{\vee} := \mathrm{Hom}_R(I, \mathrm{K}_R)$.
- Change of rings.

Unfortunately,

- $\mathrm{e}(R) = 9 \cdots$ Conjecture 1.3 is not true in general.
- $\mathrm{e}(R) = 7, 8 \cdots$ We don't know whether Conjecture 1.3 is true or not.

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Theorem 1.4 (Main Theorem)

Let R be a C-M local ring with $\dim R = 1$ and assume $\exists \mathrm{K}_R$. Let I be a faithful ideal of R.

(1) *Assume that the canonical map*

$$t : I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R) \to \mathrm{K}_R, \ x \otimes f \mapsto f(x)$$

is an isomorphism. If $r, s \geq 2$, then

$$\mathrm{e}(R) > (r+1)s \geq 6,$$

where $r = \mu_R(I)$ and $s = \mu_R(\mathrm{Hom}_R(I, \mathrm{K}_R))$.

(2) *Suppose that $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree. If $\mathrm{e}(R) \leq 6$, then $I \cong R$ or K_R.*

Higher dimensional assertion is the following.

Corollary 1.5

Let R be a C-M local ring with $\dim R \geq 1$, I a faithful ideal of R. Assume that

- *$R_{\mathfrak{p}}$ is Gorenstein, and*
- $\mathrm{e}(R_{\mathfrak{p}}) \leq 6$

for every height one prime $\mathfrak{p}$.

If $I \otimes_R \mathrm{Hom}_R(I, R)$ is reflexive, then I is principal.

Contents

Notation

In what follows, unless other specified, we assume

1. $(R, \mathfrak{m})$ a C-M local ring with $\dim R = 1$
2. $F = \mathrm{Q}(R)$ the total ring of fractions of R
3. $\mathcal{F} = \{I \mid I$ is a fractional ideal such that $FI = F\}$
4. $\exists$ a canonical module K_R of R
5. $M^\vee := \mathrm{Hom}_R(M, \mathrm{K}_R)$ for each R-module M
6. $\mu_R(M) := \ell_R(M/\mathfrak{m}M)$ for each R-module M

§2 Change of rings

Let $I \in \mathcal{F}$. Denote by

$$t : I \otimes_R I^\vee \to \mathrm{K}_R,\ x \otimes f \mapsto f(x).$$

Then the diagram

$$\begin{array}{ccc} F \otimes_R (I \otimes_R I^\vee) & \xrightarrow{\cong} & F \otimes_R \mathrm{K}_R \\ \alpha \uparrow & & \uparrow \\ I \otimes_R I^\vee & \xrightarrow{t} & \mathrm{K}_R \end{array}$$

is commutative. Hence

$$T := \mathrm{T}(I \otimes_R I^\vee) = \mathrm{Ker}\, t.$$

Lemma 2.1

$I \otimes_R I^\vee$ is torsionfree $\Longleftrightarrow$ $t : I \otimes_R I^\vee \longrightarrow \mathrm{K}_R$ is injective.

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Lemma 2.1

$I \otimes_R I^\vee$ is torsionfree $\Longleftrightarrow$ $t : I \otimes_R I^\vee \longrightarrow \mathrm{K}_R$ is injective.

We set $L = \mathrm{Im}(I \otimes_R I^\vee \xrightarrow{t} \mathrm{K}_R)$. Look at the exact sequence

$$0 \to T \to I \otimes_R I^\vee \xrightarrow{t} L \to 0.$$

Therefore we have

$$L^\vee \cong (I \otimes_R I^\vee)^\vee = \mathrm{Hom}_R(I, I^{\vee\vee}) \cong I : I =: B \subseteq F.$$

Let $R \subseteq S \subseteq B$. Then I is also a fractional ideal of S,

$$\begin{aligned} L &= L^{\vee\vee} = B^\vee = \mathrm{K}_B \subseteq S^\vee = \mathrm{K}_S \quad \text{and} \\ \mathrm{Hom}_S(I, \mathrm{K}_S) &= \mathrm{Hom}_S(I, \mathrm{Hom}_R(S, \mathrm{K}_R)) \\ &\cong \mathrm{Hom}_R(I \otimes_S S, \mathrm{K}_R) = \mathrm{Hom}_R(I, \mathrm{K}_R) = I^\vee. \end{aligned}$$

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$$\begin{aligned} L &= L^{\vee\vee} = B^\vee = \mathrm{K}_B \subseteq S^\vee = \mathrm{K}_S \quad \text{and} \\ \mathrm{Hom}_S(I, \mathrm{K}_S) &= \mathrm{Hom}_S(I, \mathrm{Hom}_R(S, \mathrm{K}_R)) \\ &\cong \mathrm{Hom}_R(I \otimes_S S, \mathrm{K}_R) = \mathrm{Hom}_R(I, \mathrm{K}_R) = I^\vee. \end{aligned}$$

Therefore the diagram

$$\begin{array}{ccc} I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) & \xrightarrow{t_S} & \mathrm{K}_S \\ \rho \uparrow & & \iota \uparrow \\ I \otimes_R I^\vee & \xrightarrow{t} & L \end{array}$$

is commutative, where $\rho(x \otimes f) = x \otimes f$.

Lemma 2.2

Let $I \in \mathcal{F}$ and $R \subseteq S \subseteq B := I : I$. If $I \otimes_R I^\vee$ is torsionfree, then

$$I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$$

is a torsionfree S-module and

$$\rho : I \otimes_R I^\vee \to I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$$

is bijective.

In particular, if $S = B$, then

$$t_B : I \otimes_B \mathrm{Hom}_B(I, \mathrm{K}_B) \to \mathrm{K}_B, \quad x \otimes f \mapsto f(x)$$

is an isomorphism of B-modules.

Proposition 2.3 (*Change of rings*)

Let $I \in \mathcal{F}$ and assume that $I \otimes_R I^\vee$ is torsionfree. If $R \subseteq \exists S \subseteq B$ such that

$$I \cong S \text{ or } \mathrm{K}_S \quad \text{as an } S\text{-module},$$

then

$$I \cong R \text{ or } \mathrm{K}_R \quad \text{as an } R\text{-module}.$$

Proof.

Suppose $I \cong S$ and consider

$$I \otimes_R I^\vee \overset{\rho}{\cong} I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) \cong \mathrm{Hom}_S(I, \mathrm{K}_S) \cong I^\vee.$$

Then $\mu_R(I) \cdot \mu_R(I^\vee) = \mu_R(I^\vee)$, so that $I \cong R$, since $\mu_R(I) = 1$. □

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Proof.

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§3 Proof of Theorem 1.4

Theorem 3.1 (Theorem 1.4)

Let I be a faithful ideal of R.

(1) *Assume that*

$$t : I \otimes_R I^\vee \to \mathrm{K}_R,\ x \otimes f \mapsto f(x)$$

is an isomorphism. If $r, s \geq 2$, then

$$\mathrm{e}(R) > (r+1)s \geq 6,$$

where $r = \mu_R(I)$ and $s = \mu_R(I^\vee)$.

(2) *Suppose that $I \otimes_R I^\vee$ is torsionfree. If $\mathrm{e}(R) \leq 6$, then $I \cong R$ or K_R.*

Proof of assertion (1) of Theorem 1.4

Choose $f \in \mathfrak{m}$ such that fR is a reduction of $\mathfrak{m}$. Let

$$S = R/fR, \quad \mathfrak{n} = \mathfrak{m}/fR \quad \text{and} \quad M = I/fI.$$

Hence

$$\mu_S(M) = r, \quad \mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) = s.$$

We write $M = Sx_1 + Sx_2 + \cdots + Sx_r$ and look at

$$(\sharp_0) \qquad 0 \to X \to S^{\oplus r} \xrightarrow{\varphi} M \to 0, \quad \varphi(\mathbf{e_i}) = x_i.$$

We get

$$(\sharp_1) \qquad 0 \to M^{\vee} \to \mathrm{K}_S^{\oplus r} \to X^{\vee} \to 0,$$

$$(\sharp_2) \qquad 0 \to \mathrm{Hom}_S(M, M) \to M^{\oplus r} \to \mathrm{Hom}_S(X, M).$$

Proof of assertion (1) of Theorem 1.4

Since $S = \mathrm{Hom}_S(M, M)$, we have by $(\sharp_2)$

$$(\sharp_3) \quad 0 \to S \xrightarrow{\psi} M^{\oplus r} \to \mathrm{Hom}_S(X, M),$$

where $\psi(1) = (x_1, x_2, \ldots, x_r)$.

By

$$(\sharp_0) \quad 0 \to X \to S^{\oplus r} \xrightarrow{\varphi} M \to 0.$$

we get

$$\ell_S(X) = r{\cdot}\ell_S(S) - \ell_S(M) = re - e = (r-1)e,$$

where $e = \mathrm{e}(R)$.

Proof of assertion (1) of Theorem 1.4

By

$$(\sharp_1) \qquad 0 \to M^\vee \to \mathrm{K}_S^{\oplus r} \to X^\vee \to 0,$$

we have

$$q := \mu_S(X^\vee) \geq \mu_S(\mathrm{K}_S^{\oplus r}) - \mu_S(M^\vee) = r{\cdot}\mu_S(\mathrm{K}_S) - \mathrm{r}_S(M).$$

Therefore

$$(r-1)e = \ell_S(X) \geq \ell_S((0) :_X \mathfrak{n}) = q \geq r^2 s - s = s(r^2 - 1).$$

Thus

$$e \geq s(r+1),$$

since $r \geq 2$.

Proof of assertion (1) of Theorem 1.4.

Suppose that $e = s(r+1)$. Then $\mathfrak{n}\cdot \mathrm{Hom}_S(X, M) = (0)$. By

$$(\sharp_3) \qquad 0 \to S \xrightarrow{\psi} M^{\oplus r} \to \mathrm{Hom}_S(X, M),$$

we have

$$\mathfrak{n}\cdot M^{\oplus r} \subseteq S\cdot(x_1, x_2, \ldots, x_r),$$

and therefore

$$\mathfrak{n}^2 M = (0).$$

Thus $\mathfrak{n}M \subseteq (0) :_M \mathfrak{n}$. Consequently

$$\begin{aligned} s = \mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) &\geq \ell_S(\mathfrak{n}M) = \ell_S(M) - \ell_S(M/\mathfrak{n}M) \\ &= e - r = s(r+1) - r. \end{aligned}$$

Hence $0 \geq rs - r = r(s-1)$, which is impossible. □

Corollary 3.2

Let R be a Gorenstein local ring with $\dim R = 1$ and $\mathrm{e}(R) \leq 6$. Let I be a faithful ideal of R. If

$$I \otimes_R \mathrm{Hom}_R(I, R) \ \text{ is torsionfree,}$$

then I is principal.

We also prove the following theorems.

Theorem 3.3

Assume that $\mathfrak{m}\overline{R} \subseteq R$. *Let* I *be a faithful fractional ideal of* R. *If* $I \otimes_R I^\vee$ *is torsionfree, then* $I \cong R$ *or* K_R.

Here $\overline{R}$ stands for the integral closure of R.

Theorem 3.4

Assume that $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$. *Let* I *be a faithful ideal of* R. *If* $I \otimes_R I^\vee \cong \mathrm{K}_R$, *then* $I \cong R$ *or* K_R.

Let k be a field.

Proposition 3.5

Let $R = k[[t^a, t^{a+1}, \dots, t^{2a-1}]]$ $(a \geq 1)$ be the semigroup ring and $I \neq (0)$ an ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Corollary 3.6

Let $R = k[[t^a, t^{a+1}, \dots, t^{2a-2}]]$ $(a \geq 3)$ be the semigroup ring and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Proof of Corollary 3.6.

Notice that R is a Gorenstein local ring with $R : \mathfrak{m} = R + kt^{2a-1}$.
Suppose that $I \ncong R$. Then $R \subsetneq B := I : I$ and therefore

$$t^{2a-1} \in B,$$

whence

$$R \subseteq S := k[[t^a, t^{a+1}, \ldots, t^{2a-1}]] \subseteq B.$$

Therefore $I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is S-torsionfree, so that

$$I \cong S \text{ or } I \cong \mathrm{K}_S$$

as an S-module by Proposition 3.5. Hence $I \cong R$ by *Change of rings*, which is contradiction. □

Remark 3.7

Corollary 3.6 gives a new class of one-dimensional Gorenstein local domains for which Conjecture 1.2 holds true.

For example, take $a = 5$. Then

$$R = k[[t^5, t^6, t^7, t^8]]$$

is Gorenstein, but not complete intersection.

Conjecture 1.2

Let R be a Gorenstein local domain with $\dim R = 1$ and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

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Conjecture 1.2

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§4 Numerical semigroup rings

Setting 4.1

Let $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$.

We set

- $H = \langle a_1, a_2, \ldots, a_\ell \rangle := \{\sum_{i=1}^{\ell} c_i a_i \mid 0 \le c_i \in \mathbb{Z}\}$
- $R = k[[H]] := k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \ \subseteq \ V = k[[t]]$
- $\mathfrak{m} = (t^{a_1}, t^{a_2}, \ldots, t^{a_\ell})$ the maximal ideal of R
- $c = \mathrm{c}(H) := \max(\mathbb{Z} \setminus H) + 1$ the conductor of H
- $\mathfrak{c} := R : V = t^c V$

Notice that

- R is a C-M local domain with $\dim R = 1$ and $V = \overline{R}$.
- $\mathrm{e}(R) = a_1 = \mu_R(V)$.

Definition 4.2

Let $I \in \mathcal{F}$. Then I is said to be *a monomial ideal*, if

$$I = \sum_{n \in \Lambda} Rt^n$$

for some $\Lambda \subseteq \mathbb{Z}$.

Set

$$\mathcal{M} = \{I \in \mathcal{F} \mid I \text{ is a monomial ideal}\}.$$

Notice that

- R is a C-M local domain with $\dim R = 1$ and $V = \overline{R}$.
- $\mathrm{e}(R) = a_1 = \mu_R(V)$.

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Let $I \in \mathcal{F}$. Then I is said to be *a monomial ideal*, if

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for some $\Lambda \subseteq \mathbb{Z}$.

Set

$$\mathcal{M} = \{I \in \mathcal{F} \mid I \text{ is a monomial ideal}\}.$$

Passing to the monomial ideal $t^{-q}I$ for some $q \in \mathbb{Z}$, we may assume

$$R \subseteq I \subseteq V.$$

A canonical ideal K_R of R is given by

$$\mathrm{K}_R = \sum_{n \in \mathbb{Z} \setminus H} Rt^{a-n}$$

where $a = \mathrm{c}(H) - 1$ $(= \max(\mathbb{Z} \setminus H))$. Therefore

$$a - n \notin H \iff t^n \in \mathrm{K}_R$$

for $\forall n \in \mathbb{Z}$.

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$$a - n \notin H \iff t^n \in \mathrm{K}_R$$

for $\forall n \in \mathbb{Z}$.

From now on, we assume that $\mathrm{e}(R) = a_1 \geq 2$. Set

$$\alpha_i = \max\{n \in \mathbb{Z} \setminus H \mid n \equiv i \mod e\} \quad (0 \leq i \leq e-1)$$

and

$$\mathcal{S} = \{\alpha_i \mid 1 \leq i \leq e-1\}.$$

Hence

$$\alpha_0 = -\mathrm{e}(R),\ \sharp\mathcal{S} = \mathrm{e}(R) - 1,\ a = \max \mathcal{S} \ \text{ and } \ \alpha_i \geq i \ \ (1 \leq \forall i \leq e-1).$$

Fact 4.3

- $\mathrm{K}_R = \sum_{s \in \mathcal{S}} Rt^{a-s}$.
- $\{t^{a-s} \mid s \in \mathcal{S} \text{ s.t. } t^s \in R : \mathfrak{m}\}$ *forms a minimal system of generators for* K_R.

Example 4.4

Let $H = \langle 7, 8, 10 \rangle$. The figure of H is the following (the gray part).

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28			$\cdots$			

We have $c = \mathrm{c}(H) = 20$, $a = 19$,

$$\begin{aligned} \mathcal{S} &= \{1, 9, 3, 11, 19, 13\}, \text{ and} \\ a - \mathcal{S} &= \{18, 10, 16, 8, 0, 6\}. \end{aligned}$$

Therefore

$$\mathrm{K}_R = (t^{18}, t^{10}, t^{16}, t^{8}, 1, t^{6}) = (1, t^{6}).$$

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Therefore

$$\mathrm{K}_R = (t^{18}, t^{10}, t^{16}, t^8, 1, t^6) = (1, t^6).$$

The goal of this section is the following.

Theorem 4.5

Let $b = \min \mathcal{S}$ and suppose $t^b \in R : \mathfrak{m}$. Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. If $I \otimes_R I^\vee \cong \mathrm{K}_R$, then $I \cong R$ or K_R.

Corollary 4.6

Suppose that $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$. Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. If $I \otimes_R I^\vee \cong \mathrm{K}_R$, then $I \cong R$ or $I \cong \mathrm{K}_R$.

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Suppose that $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$. *Let* $I \in \mathcal{M}$ *such that* $R \subseteq I \subseteq V$. *If* $I \otimes_R I^\vee \cong \mathrm{K}_R$, *then* $I \cong R$ *or* $I \cong \mathrm{K}_R$.

Example 4.7

Let $H = \langle 7, 22, 23, 25, 38, 40 \rangle$. Then $\mathcal{S} = \{15, 16, 18, 33, 41\}$. We have $a = 41, b = 15$, and

$$\mathfrak{m}{\cdot}t^{15} \subseteq R,$$

but

$$\mu_R(\mathfrak{m}) = 6 < \mathrm{e}(R) = 7.$$

§5 The case where $\mathrm{e}(R) = 7$

In this section, we maintain Setting 4.1.

Lemma 5.1

Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. Suppose that

$$\mu_R(I) = \mu_R(J) = 2, \quad IJ = \mathrm{K}_R \quad \textit{and} \quad \mu_R(\mathrm{K}_R) = 4.$$

Then

$$\mathrm{e}(R) = a_1 \geq 8,$$

where $J = \mathrm{K}_R : I$.

Theorem 5.2

Suppose that $\mathrm{e}(R) = a_1 \leq 7$. *Let* $I \in \mathcal{M}$. *If* $I \otimes_R I^\vee$ *is torsionfree, then* $I \cong R$ *or* K_R.

Proof.

We may assume that $I \otimes_R I^\vee \cong \mathrm{K}_R$. Suppose that $I \ncong R$ and $I \ncong \mathrm{K}_R$. Then

$$4 \leq \mu_R(I)\cdot\mu_R(I^\vee) = \mu_R(\mathrm{K}_R) = \mathrm{r}(R) \leq \mathrm{e}(R) - 1 \leq 6.$$

If $\mathrm{r}(R) = 6$, then $\mathrm{r}(R) = \mathrm{e}(R) - 1$. Thus

$$\mu_R(\mathfrak{m}) = \mathrm{e}(R) = 7$$

which is contradiction. Hence $\mathrm{r}(R) = 4$, so that $\mu_R(I) = \mu_R(I^\vee) = 2$ which violates Lemma 5.1. □

Corollary 5.3

Suppose that $R = k[[H]]$ *is Gorenstein with* $\mathrm{e}(R) \leq 7$*. Let* $I \in \mathcal{M}$*. If*

$$I \otimes_R \mathrm{Hom}_R(I, R) \text{ is torsionfree,}$$

then I *is principal.*

§6 How to compute the torsion part $\mathrm{T}(I \otimes_R J)$

Let $(R, \mathfrak{m})$ be a C-M local ring with $\dim R = 1$. Let

$$I = (1, f) \ (= R + Rf) \in \mathcal{F}$$

where $f \in F \setminus R$.

Theorem 6.1

Let $J \in \mathcal{F}$. Then

$$(J : I)/(R : I)J \cong \mathrm{T}(I \otimes_R J), \quad \overline{c} \longmapsto f \otimes c - 1 \otimes cf$$

as an R-module.

§6 How to compute the torsion part $\mathrm{T}(I \otimes_R J)$

Let $(R, \mathfrak{m})$ be a C-M local ring with $\dim R = 1$. Let

$$I = (1, f) \ (= R + Rf) \in \mathcal{F}$$

where $f \in F \setminus R$.

Theorem 6.1

Let $J \in \mathcal{F}$. Then

$$(J : I)/(R : I)J \cong \mathrm{T}(I \otimes_R J), \quad \overline{c} \longmapsto f \otimes c - 1 \otimes cf$$

as an R-module.

§7 Examples

Let $I \in \mathcal{M}$. We set $J = \mathrm{K}_R : I$.

Condition : $IJ = \mathrm{K}_R$ and $\mu_R(\mathrm{K}_R) = 4$

Example 7.1

Let $R = k[[t^8, t^{11}, t^{14}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. We take $I = (1, t)$ and set $J = \mathrm{K}_R : I$. Then

$$J = (1, t^3), \ IJ = \mathrm{K}_R \text{ and } \mu_R(\mathrm{K}_R) = 4,$$

but

$$\mathrm{T}(I \otimes_R J) \cong R/\mathfrak{m}.$$

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Let $I \in \mathcal{M}$. We set $J = \mathrm{K}_R : I$.

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but

$$\mathrm{T}(I \otimes_R J) \cong R/\mathfrak{m}.$$

Proof of Example 7.1

The figure of $H = \langle 8, 11, 14, 15 \rangle$ is the following.

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32				$\cdots$			

We have $\mathrm{c}(H) = 22$, $a = 21$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. Let $I = (1, t)$. Then

$$\begin{aligned} J &= \mathrm{K}_R : I = (t^n \mid n \in \mathbb{Z} \text{ s.t. } 21 - n, 20 - n \notin H) \\ &= (1, t^3). \end{aligned}$$

Hence $IJ = \mathrm{K}_R$, $\mu_R(I) = \mu_R(J) = 2$ and $\mu_R(\mathrm{K}_R) = 4$.

Proof of Example 7.1

0	1	2	3	4	5	6	7
8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
32				$\cdots$			

Since $R : I = (t^n \mid n \in H, n+1 \in H) = (t^{14}, t^{15}, t^{24}, t^{27})$, we get

$$(R : I)J = (t^{14}, t^{15}, t^{17}, t^{18}, t^{24}, t^{27}).$$

On the other hand,

$$\begin{aligned} J : I &= (t^n \mid 21 - n, 20 - n, 19 - n \notin H) \\ &= (t^{14}, t^{15}, \underline{t^{16}}, t^{17}, t^{18}). \end{aligned}$$

Hence $t^{16} \notin (R : I)J$ and $\mathfrak{m} \cdot t^{16} \subseteq (R : I)J$. Thus

$$\mathrm{T}(I \otimes_R J) \cong (J : I)/(R : I)J = R\overline{t^{16}} \cong R/\mathfrak{m}.$$

$\square$

Remark 7.2

In the ring R of Example 7.1 $\nexists$ monomial ideals I such that $I \ncong R, I \ncong \mathrm{K}_R$, and $I \otimes_R I^\vee$ is torsionfree.

The following ideals also satisfy

$$IJ = \mathrm{K}_R \text{ and } \mu_R(\mathrm{K}_R) = 4$$

but $I \otimes_R I^\vee$ is not torsionfree.

- $H = \langle 8, 9, 10, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
- $H = \langle 8, 11, 12, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
- $H = \langle 8, 11, 14, 23 \rangle, \mathrm{K}_R = (1, t^3, t^9, t^{12}), I = (1, t^3)$.
- $H = \langle 8, 13, 17, 18 \rangle, \mathrm{K}_R = (1, t, t^5, t^6), I = (1, t)$.
- $H = \langle 8, 13, 18, 25 \rangle, \mathrm{K}_R = (1, t^5, t^7, t^{12}), I = (1, t^5)$.

If $\mathrm{e}(R) \geq 9$, then Conjecture 1.3 is not true in general.

Example 7.3

Let $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. Let $I = (1, t)$ and put $J = \mathrm{K}_R : I$. Then

$$J = (1, t^3), \quad \mu_R(I) = \mu_R(J) = 2 \text{ and } \mu_R(\mathrm{K}_R) = 4,$$

but $I \otimes_R I^\vee$ is torsionfree.

Thank you very much for your attention.

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